Exam 1

Graded

Student

Alejandro De La Torre

Total Points

27 / 50 pts

Question 1

1a **1** / 1 pt

✔ **+ 1 pt** Correct

Question 2

1b **1** / 1 pt

✔ **+ 1 pt** Correct or correct with respect to part (a)

Question 3

1c **1** / 1 pt

✔ **+ 1 pt** Correct or correct with respect to part (a)

Question 4

1d **1** / 1 pt

✔ **+ 1 pt** Correct or correct with respect to part (a)

Question 5

1e **1** / 1 pt

✔ **+ 1 pt** Correct or correct with respect to part (a)

Question 6

2a **0** / 1 pt

✔ **+ 0 pts** Incorrect or incomplete

Question 7

2b **0** / 1 pt

✔ **+ 0 pts** Incorrect or incomplete

Question 8

2c **0** / 1 pt

✔ **+ 0 pts** Incorrect or incomplete

Question 9

2d

1 / 1 pt

✔ **+ 1 pt** Correct

Question 10

2e

1 / 1 pt

✔ **+ 1 pt** Correct

Question 11

3

5 / 10 pts

✔ **+ 5 pts** There is evidence of a reasonable basic proof outline, but there are multiple flaws in the argument.

1. The real numbers are closed under addition, but this does not mean that the irrational numbers are.
2. A proof by contradiction does not begin with an if...then statement. It begins with new hypotheses: the original hypotheses and the negation of the original conclusion.
3. Since we have proven in class that sqrt(2) is irrational and the sum of a rational plus an irrational number is irrational.
4. This is not the correct negation of the given conclusion.

Question 12

4

5 / 10 pts

✔ **+ 5 pts** There is evidence of a reasonable basic proof outline, but there are multiple flaws in the argument.

5. Base case is n=3.
6. For some.
7. You must prove this.

Question 13

5

3 / 10 pts

✔ **+ 3 pts** An attempt was made, but proof has major errors or inconstancies.

8. This is what you are trying to prove.
9. Why is this true?
10. The proof is incomplete.

Question 14

6

7 / 10 pts

✔ **+ 7 pts** Almost correct, but has a major flaw.

11 This is not exactly the distance you are looking for.

12 N_1 and N_2 do not necessarily equal epsilon/2.

Math 421 Fall 2025

Exam 1

Name: Alejandro De la Torre

The exam contains 7 pages (including this cover page) and 6 questions. The total number of possible points is 50. You have 90 minutes to complete the exam.

- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **No outside resources allowed.** You may not use any notes, calculators, textbooks, or other resources on this exam. Make sure all electronic devices are silenced and put away.
- **Partial credit will be given,** so write down as much as you know even if some part of the proof might be missing.
- **Read the questions carefully,** make sure you answer the question being asked and understand the statement of the problem.
- **Answer all questions in the space provided,** you may use the back of pages as scrap paper, but only what is written under the question will be graded.

Do not write in the table to the right.

Question	Points	Score
1	5	
2	5	
3	10	
4	10	
5	10	
6	10	
Total:	50	

1. (5 points) Consider the following subsets of $\mathbb{R}$ and use them to answer the following questions. Write your answers in the boxes provided. **No justification is needed.**

$|x-2| < 3 = \begin{cases} x-2 < 3 \Rightarrow x < 5 \\ -(x-2) < 3 \quad x < 1 \end{cases}$

- $A = \{x \in \mathbb{R} : |x-2| < 3\} \Rightarrow$

  $\Rightarrow A = (-1, 5)$

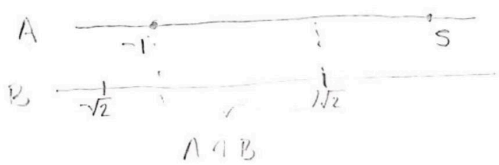
$x = 0 \Rightarrow 2 < 3$
$x = 1 \Rightarrow 1 < 3$ $\Rightarrow -1$ lower bound
$x = 2 \Rightarrow 0 < 3$
$x = 3 \Rightarrow 1 < 3$
$x = 4 \Rightarrow 2 < 3$
$x = 5 \Rightarrow 3 \not< 3 \Rightarrow 5$ upper bound and inf
$6 \Rightarrow 4 \not< 3$
$x = -1 \Rightarrow$

- $B = \{x \in \mathbb{R} : x^2 \leq 2\}$

  $\Rightarrow [-\sqrt{2}, \sqrt{2}]$

(a) Express $A \cap B$ as an interval in $\mathbb{R}$.

or



$\boxed{(-1, \sqrt{2}]}$

Find the following if they exist or indicate they do not exist:

= DNE

(b) $\min(A \cap B)$

b/c -1 is not included

- lower bound can be -1 to $-\infty$
- the greatest lower bound can be -1

$\boxed{\text{DNE}}$

(c) $\max(A \cap B)$

$\boxed{\sqrt{2}}$

(d) $\inf(A \cap B)$

greatest lower bound

$\boxed{-1}$

(e) $\sup(A \cap B)$

$\boxed{\sqrt{2}}$

2. (5 points) Determine if each of the following statements is TRUE or FALSE by completely filling in the corresponding bubble. **No justification is needed.**

(a) The negation of the statement "x and y are both even integers" is "x and y are both odd integers".

→ or

x is even and y is odd

x is odd and y is odd

● TRUE

○ FALSE

→ If x & y are both not even integers then they must be both odd

(b) All unbounded sequences are divergent.

All convegent sequences are bounded

○ TRUE

All divegent sequences are unbounded → not sure the other way around

→ ● FALSE

(c) All divergent sequences are unbounded.

● TRUE

○ FALSE

$\forall x \in S$, $\max(S) \geq x$

(d) If a set S has a maximum value, then $\sup S \in S$.

● TRUE

○ FALSE

(e) The negation of the statement "for all real numbers a, there exists a real number b such that $ab = 1$" is the statement "there exists a real number a such that for all real numbers b, $ab \neq 1$".

negation ≠ converse

→ for all real numbers 'a', there exists a real number b such that, ab≠1

there exists a real number a such that for all real numbers b, ab=1

● TRUE

○ FALSE

maybe this directly negates it → no, if that a really exists such that for all real numbers b, ab≠1 then that would dis-truth the original statement

3. (10 points) Use a **proof by contradiction** to prove the following statement. If you do any scratch work on this page you should clearly indicate what should be graded as your final proof. Your proof should be clearly laid out and your logic should be easy to follow.

Prove: If $x \in \mathbb{R}$, then $x + \sqrt{2}$ is irrational or $x - \sqrt{2}$ is irrational.

We argue by contradiction. Assume by way of contradiction that if $x \in \mathbb{R}$ then $x + \sqrt{2}$ is rational such that $x + \sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$x + \sqrt{2} = \frac{p}{q} \Rightarrow x = \frac{p}{q} - \sqrt{2}$$

Then x is irrational as we know $\sqrt{2}$ is irrational. Thus this is a contradiction. Since $\mathbb{R}$ is closed under addition, it follows that if $x \in \mathbb{R}$, then $x - \sqrt{2}$ too is irrational as if too, if we assume x is rational by way of contradiction, then $x = \frac{p}{q} + \sqrt{2}$ would reveal x to be irrational, rendering the same contradiction.

Scratch Work

$(x+\sqrt{2})(x+\sqrt{2})$

$x^2 + x\sqrt{2} + x\sqrt{2} + 2$

$\rightarrow x^2 + 2\sqrt{2}x + 2$

for $(x+\sqrt{2})^2 = \frac{p^2}{q^2}$.

I know in class Prof. Work mention we can say $\sqrt{2}$ is irrational unless explicitly asked to prove it. Here, to say $\frac{p}{q} - \sqrt{2}$ is irrational because $\sqrt{2}$ is irrational is sorta a risk but I'll take it.

4. (10 points) Use a **proof by induction** to prove the following statement. If you do any scratch work on this page you should clearly indicate what should be graded as your final proof. Your proof should be clearly laid out and your logic should be easy to follow.

Prove: If $n \in \mathbb{N}$ and $n \geq 3$, then $2^n > n + 4$.

We prove by induction. If $n \in \mathbb{N}$ and $n \geq 3$

Base case: P_n: $2^n > n + 4$

$2^3 = 8 \cdot 2$
16

If $n = 4$ then it follows that,

$2^4 > (4) + 4$

$\Rightarrow 16 > 8$

Inductive Step:

Assume by way of induction that $2^n > n+4$ for $n \in \mathbb{N}$, $n \geq 3$, then

$2^{(n+1)} > (n+1)+4$

$\Rightarrow 2^n \cdot 2 > n + 4 + 1$

(this is literally P_n: $2^n > n+4$)

Thus "since $n+1$ was arbitrary" and $\mathbb{N}$ is closed under multiplication and addition then if $n \in \mathbb{N}$ $n \geq 3$, then $2^n > n+4$. $\square$

Scratch Work

P_n: $2^n > n+4$

$\Rightarrow$ we want to show that for P_{n+1} that

$2^{(n+1)} > (n+1)+4$

Following the N5 peano axiom of the $\mathbb{N}$ that forms the basis of induction.

$(n+1) \geq 3$ right for

$2^n > n+4$? no no way - stick to what you know that is a given truth

So basically we want to show that $n+1$ follows from n case where we multiply RHS by 2 and add 1 on RHS

$\sqrt{1-\frac{1}{n}}$

5. (10 points) Prove, using the **precise definition for the limit of a sequence. Do NOT use any of the algebraic limit theorems or any previous homework problems.**, that

$n = 100 \Rightarrow \sqrt{1+\frac{1}{100}} = 1$

$n \to \infty = 0 \Rightarrow \sqrt{1} = 1$

$$\lim \sqrt{1+\frac{1}{n}} = 1.$$

Please write your complete proof on the bottom half of the page labeled "**Proof.**", only work that appears there will be graded. Your proof should be clearly laid out and your logic should be easy to follow. You may use the top half of the page labeled "*Discussion.*" for any scratch work. I may look at the scratch work if your proof is incorrect to help maximize the partial credit given.

Discussion.

$\forall \varepsilon > 0\ \exists N \mid n > N \Rightarrow |S_n - S| < \varepsilon \quad ; \quad S_n = \sqrt{1+\frac{1}{n}}$

$\Rightarrow \left|\sqrt{1+\frac{1}{n}} - 1\right| < \varepsilon$

$\Rightarrow \left|\left(1+\frac{1}{n}\right)^{\frac{1}{2}} - 1\right| < \varepsilon$

$\Rightarrow \left|\frac{1}{(1+\frac{1}{n})^2} - \frac{(1+\frac{1}{n})^2}{(1+\frac{1}{n})}\right| < \varepsilon$

wait

$\left|\sqrt{1+\frac{1}{n}} - 1\right| < \varepsilon$

$\Rightarrow \left|\frac{\sqrt{1+\frac{1}{n}}}{1} - \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}}\right| < \varepsilon$

$\Rightarrow \left|\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}} - \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}}\right| < \varepsilon$

or mult. by its conjugate:

$\Rightarrow \left|\sqrt{1+\frac{1}{n}} \cdot \frac{\sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}} - 1\right| < \varepsilon$

$\Rightarrow \left|\frac{(1+\frac{1}{n})(1-\frac{1}{n})}{\sqrt{1-\frac{1}{n}}} - 1\right| < \varepsilon$

$\Rightarrow \left|\frac{1-\frac{1}{n^2}}{\sqrt{1-\frac{1}{n}}} - 1\right| < \varepsilon$

$\Rightarrow \left|\frac{1-\frac{1}{n^2}}{\sqrt{1-\frac{1}{n}}} - \frac{\sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}\right| < \varepsilon$

$\Rightarrow \left|\frac{1-\frac{1}{n^2} - \sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}\right| < \varepsilon$

$\Rightarrow \left|\frac{\frac{n^2}{n^2} - \frac{1}{n^2} - \sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}\right| < \varepsilon$

$\Rightarrow \left|\frac{\frac{n^2-1}{n^2} - \sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}\right| < \varepsilon$

☆ We can say

$\left|\frac{1-\frac{1}{n^2} - \sqrt{1-\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}\right| \le \frac{1}{\sqrt{1-\frac{1}{n}}} < \varepsilon$

$\Rightarrow \sqrt{1-\frac{1}{n}} > \frac{1}{\varepsilon}$

$\Rightarrow 1-\frac{1}{n} > \frac{1}{\varepsilon^2}$

$\Rightarrow -\frac{1}{n} > \frac{1}{\varepsilon^2} - 1$

$\Rightarrow \frac{1}{n} < 1 - \frac{1}{\varepsilon^2}$

$\Rightarrow n > \frac{1}{1-\frac{1}{\varepsilon^2}}$

since $n > N$

Proof.

Let $\varepsilon > 0$. There exists an $N = \frac{1}{1-\frac{1}{\varepsilon^2}}$ such that $n > N$, then

$\left|\sqrt{1+\frac{1}{n}} - 1\right| < \varepsilon \Rightarrow \left|\sqrt{1+\frac{1}{n}} - \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}}\right| < \varepsilon$

8

9

$\left|\frac{2\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}} - \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}}\right| = \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}} < \varepsilon$

$\Rightarrow \frac{2\sqrt{1+\frac{1}{n}} - \sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}} < \varepsilon \Rightarrow \frac{1}{1}$

10

Since $\varepsilon > 0$, then $\lim \sqrt{1+\frac{1}{n}} = 1$

$\sqrt{1+\frac{1}{(\frac{1}{1-\frac{1}{\varepsilon^2}})}} - 1$

$1. \frac{1}{1-\frac{1}{\varepsilon^2}} =$

$\sqrt{1+1-\frac{1}{\varepsilon^2}} - 1$

$\forall \varepsilon > 0 \ \exists N_1 \mid n > N \Rightarrow |s_n - s| < \varepsilon$

6. (10 points) Prove the following statement, using the **precise definition for the limit of a sequence. Do NOT use any of the algebraic limit theorems or any previous homework problems.** If you do any scratch work on this page you should clearly indicate what should be graded as your final proof. Your proof should be clearly laid out and your logic should be easy to follow.

Prove: If $\lim s_n = c$ and $\lim t_n = c$, then $\lim |s_n - t_n| = 0$. this is new.

Let $\varepsilon > 0$. If $\lim s_n = c$ then there exists $N_1 = \frac{\varepsilon}{2}$ such that $n > N_1$ then

$$|s_n - s| < \frac{\varepsilon}{2}$$

Notice that $|s_n - t_n - (c - c)| \leq |s_n - c| + |t_n - c| < \varepsilon$. Then if $\lim t_n = c$ there exists $N_2 = \frac{\varepsilon}{2}$ such that $n > N_2$ then

$$|t_n - t| < \frac{\varepsilon}{2}$$

Let $n > \max\{N_1, N_2\}$ then $|s_n - c| + |t_n - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary then $\lim |s_n - t_n| = 0$. $\square$

(Scratch work)

notice $|(s_n - t_n) - (c - c)| \leq |s_n - c| + |t_n - c| < \varepsilon$

by triangle inequality

so we want to find N_1 and N_2 such that $n > \max\{N_1, N_2\}$

$|s_n - t_n - c + c| \leq |s_n - c| + |t_n - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus since $\varepsilon > 0$ was arbitrary we can then conclude that $\lim |s_n - t_n| = 0$

$|s_n - c| + |t_n - c| < \varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow$